A FEW EXERCISES WITH THE LIFE TABLE FUNCTIONS

S. Mitra, Emory University

Abstract

It has been observed that the average age of a stationary population is equal to the average number of years that remains to be lived by that population. This age or this number of years has been shown as approximately equal to the age at which the life expectancy is also equal to that age in number of years. Interestingly enough, this age can be obtained as the x coordinate of the point of intersection of the curves of the functions $x\ell_x$ and T_x. Also, the tangent to the curve of T_x, at this intersection, crosses the x axis at a point which corresponds to twice the average age of the stationary population. A biproduct of this investigation is an interesting boundary condition for the ratio of the expectation of life at birth to the average age. This ratio has been found to lie between $2\ell_{\bar{x}}$ and $2(\ell_{\bar{x}})^{\frac{1}{2}}$, where $\ell_{\bar{x}}$ is the proportion surviving from birth to the average age $\bar{x}$.

1. Definitions and Properties of Life Table Functions

The principal life table functions are conveniently defined as

(a) ℓ_x, the proportion of survivors from birth to age x, such that $\ell_0 = 1$;

(b) $\mu_x = -(1/\ell_x)\ (d\ell_x/dx)$, the force of mortality at age x;

(c) $T_x = \int_x^\alpha \ell_a da$, where α is the upper age limit at which $\ell_x = 0$;

(d) $\overset{0}{e}_x = T_x/\ell_x$ is the expectation of life at age x.

The following properties of the functions may be noted:

(a') $d\ell_x/dx$ is uniformly negative, that is to say, ℓ_x is a monotonically declining function. Further, the curve of ℓ_x has a point of inflection, i.e., $d^2\ell_x/dx^2$ equals zero for some $x = x'$. This results from the lower risk of death usually in the age interval 10-15, compared to other ages (Mitra, 1973).

(b') The force of mortality assumes its lowest value, that is, $d\mu_x/dx = 0$ for some $x = \tilde{x}$, also in the age interval 10-15 (Mitra, ibid.).

(c') $dT_x/dx = -\ell_x$.

(d') $\overset{0}{e}_x$ assumes its maximum value, that is, $d\overset{0}{e}_x/dx = 0$ for some $x = \hat{x}$, usually in the age interval 0-5.

From (d), $T_x = \ell_x\overset{0}{e}_x$, so that

$$dT_x/d_x = -\ell_x\ (-1/\ell x)(d\ell x/dx)\overset{0}{e}_x + \ell_x d\overset{0}{e}_x/dx$$

$$= -\ell_x(\overset{0}{e}_x\mu_x - d\overset{0}{e}_x/dx) \qquad (1)$$

Equating (1) with (c') and solving

$$d\overset{0}{e}_x/dx = \overset{0}{e}_x\mu_x - 1 \qquad (2)$$

The maximum life expectancy corresponds to age $\hat{x}$, where,

$$\overset{0}{e}_{\hat{x}} = 1/\mu_{\hat{x}} \qquad (3)$$

It was shown earlier (Mitra, ibid.) that $\hat{x} < \tilde{x} < x'$.

2. Average Age of a Stationary Population

Interpreting the life table functions $\ell_x dx$ and T_x as the population sizes in the age intervals x to x+dx and ages x and above respectively in the corresponding stationary population, the average age can be defined as (Mitra, 1965)

$$m = \int_0^\alpha x\ell_x dx \Big/ \int_0^\alpha \ell_x dx = \int_0^\alpha x\ell_x dx/T_0 \qquad (4)$$

Integrating by parts, the integral

$$\int_0^\alpha x\ell_x dx = -\int_0^\alpha x dT_x = xT_x\Big]_0^\alpha + \int_0^\alpha T_x dx = \int_0^\alpha T_x dx \qquad (5)$$

as $T_\alpha = 0$. Since $T_x = \ell_x\overset{0}{e}_x$, (5) can be re-written as

$$\int_0^\alpha x\ell_x dx = \int_0^\alpha T_x dx = \int_0^\alpha \ell_x\overset{0}{e}_x dx \qquad (6)$$

Accordingly, (4) can be expressed in two different ways, namely,

$$m = \int_0^\alpha x\ell_x dx/T_0 = \int_0^\alpha \ell_x\overset{0}{e}_x dx/T_0 \qquad (7)$$

The last expression in (7) can be interpreted as the average number of years that remains to be lived in the stationary population. Thus, the total population T_0, that on an average, is m years old, can expect to live for an additional m number of years, again from the perspective of an arithmetic average.

3. Expectation of Life at Age m

Although, the average age of the population is m and so is the average life expectancy, it does not follow that the life expectancy at the exact age m is also m, although, intuitively, that seems to suggest itself. It can, however, be shown that, under certain simplifying assumptions, the equality of the two is, at least, a reasonable approximation. The assumptions are the same as those that are made to derive the expected value of a function of a variable, namely, that, the function has a Taylor's expansion and further, the higher moments or higher derivatives or both of the variable concerned, are relatively small. In other words, assuming Taylor's expansion of a function f(x) at $x = m = E(x)$, and taking expected values,

$$E[f(x)] = f(m) + E \sum_{r=2}^{\infty} \frac{(x-m)^r}{r!} f^r(x)_{x=m} \quad (8)$$

since $E(x-m) = 0$. Further, when second and higher order moments or the second and higher order derivatives or both are small,

$$E[f(x)] = f(m) \quad (9)$$

It is well known that $\overset{0}{e}_x$ is a continuous and a smooth function of x, and hence the assumption of Taylor's expansion can be held. Also the derivative of $\overset{0}{e}_x$, decreases smoothly from a value of 0 at maximum life expectancy to a value, no smaller than -1 at the last age (see equation 2), because $\overset{0}{e}_x \mu_x$ is nonnegative in the entire age range. Accordingly, the second derivative is quite small (of the order of $1/\alpha$) and the higher order derivatives are even smaller. Thus, the second term of (8), namely, $\frac{V(x)}{2!} f''(x)_{x=m}$ is expected to be small for $f(x) = \overset{0}{e}_x$ and therefore, can be neglected for all practical purposes. The higher order terms will be even smaller and therefore,

$$E(\overset{0}{e}_x) = m \approx (\overset{0}{e}_x)_{x=m} = \overset{0}{e}_m \quad (10)$$

where ≈ stands for approximate equality. Alternatively, the average age can be approximately determined as the x or the y coordinate of the point of intersection of the line y = x and the curve $y = \overset{0}{e}_x$ (at which $\bar{x} = \overset{0}{e}_{\bar{x}}$). These parameters obtained from appropriate life tables (Keyfitz and Flieger, 1971) are presented in Table 1 for a few countries, covering a wide range of variation of patterns of mortality.

It may be noted that m is uniformly greater than $\bar{x}$ by about half a year for all mortality levels. Part of this bias must have resulted from the computational procedure based on the approximate equality

$$\int_a^{a+5} T_x dx \approx \frac{5}{2} (T_a + T_{a+5}) \quad (11)$$

in which, the right hand side uniformly overestimates the integral because the curve of T_x is concave upwards. In any event, the empirical demonstration of the near equality of $\bar{x}$ and m, verifies the theoretical formulation arrived at in expression (10).

4. The Curves of T_x and $x\ell_x$

The definite integrals of the functions T_x and $x\ell_x$ over the entire age range were earlier shown to be equal (see expression 6), however, the curves that they generate are quite different and have some interesting features.

The derivative of T_x is $-\ell_x$ which is uniformly negative, assumes the value of -1 at age 0, slowly decreases thereafter and becomes equal to 0 at the highest age α. Simultaneously, the function T_x declines to zero from a maximum value of T_0.

The function $x\ell_x$, on the other hand, assumes its lowest value of zero at both extremes of the age interval. Its derivative

$$\frac{d}{dx} (x\ell_x) = \ell_x(1-x\mu_x) \quad (12)$$

indicates a maximum value of the function at some $x = x''$ where

$$x'' = 1/\mu_{x''} \quad (13)$$

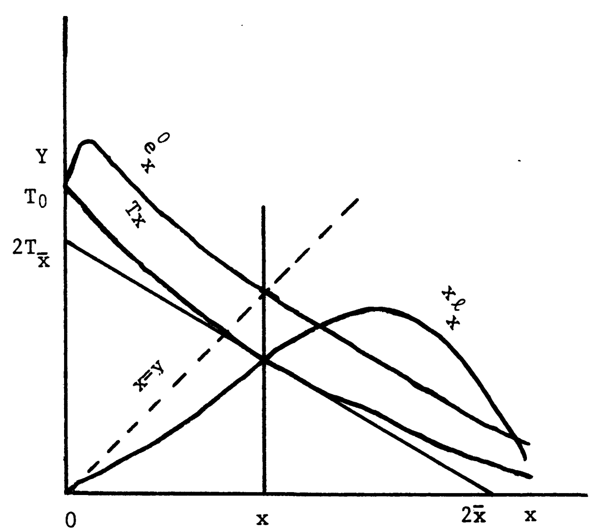

Figure 1. The Curves of $x\ell_x$ and T_x

at which the function $x\ell_x$ assumes its maximum value of $\ell_{x''}/\mu_{x''}$. Because of the nature of the functions ℓ_x and μ_x, the function $x\ell_x$ has only one maximum and thus the curve of the function is bell shaped.

The general nature of the curves of T_x and $x\ell_x$ may be studied from Figure 1.

a. The Point of Intersection

The point of intersection between $x\ell_x$ and T_x has an interesting feature. At this point,

$$T_x = x\ell_x \text{ or } x = \overset{0}{e}_x \tag{14}$$

and as was shown earlier, this happens at $x = \bar{x}$ where $\bar{x}$ is a close approximation of the average age m of the stationary population.

It can be shown that in most cases, $x < x''$ where the latter age corresponds to the maximum value of $x\ell_x$. This is so because, in the neighborhood of age $\bar{x}$, the life expectancy is usually a declining function of age so that, its derivative (see expression 2) $\overset{0}{e}_x\mu_x - 1$ is negative at $x = \bar{x}$. Since $\bar{x} = \overset{0}{e}_x$,

$$\overset{0}{e}_{\bar{x}}\mu_{\bar{x}} - 1 = \bar{x}\mu_{\bar{x}} - 1 < 0 \tag{15}$$

so that (12) is positive at $x = \bar{x}$ and therefore, $\bar{x} < x''$.

b. The Tangent to T_x

The tangent to T_x meets the x axis at a point, the x coordinate of which is (because the derivative of T_x is $-\ell_x$),

$$x + T_x/\ell_x = x + \overset{0}{e}_x \tag{16}$$

Therefore, the tangent at $T_{\bar{x}}$ is $\bar{x} + \overset{0}{e}_{\bar{x}} = 2\bar{x}$ which must intersect the y axis at $2T_{\bar{x}}$. Since, the curve of $T_{\bar{x}}$ is concave upwards.

$$2T_{\bar{x}} < T_0 \tag{17}$$

(col. 6, Table 1), which means that more than half of the population is less than $\bar{x}$ years old. In other words, the median age is less than the average age, an expected result in view of the shape of T_x.

Incidentally, the area of the right triangle with sides T_0 and $2\bar{x}$ is also a close approximation of the area of the curve bounded by T_x as well as that by $x\ell_x$ over the age interval, $(0,\alpha)$. Also, the area of the right triangle with the tangent at (x, T_x) as the hypotenuse and, therefore, sides of lengths $x + \overset{0}{e}_x$ and $\ell_x(x+\overset{0}{e}_x)$ is given by

$$A(x) = \frac{(x\ell_x + T_x)(x + \overset{0}{e}_x)}{2} \tag{18}$$

At $x = 0$,

$$A(0) = \frac{1}{2} T_0^2 \tag{19}$$

Now,

$$\frac{dA(x)}{dx} = \frac{1}{2} [x\frac{d\ell_x}{dx} (x+\overset{0}{e}_x) + (x\ell_x+T_x)\overset{0}{e}_x\mu_x] \tag{20}$$

$$= \frac{1}{2} [-x\ell_x\mu_x(x+\overset{0}{e}_x) + (x\ell_x+T_x)\overset{0}{e}_x\mu_x] \tag{21}$$

$$= \frac{1}{2} \ell_x\mu_x(\overset{0}{e}{}_x^2 - x^2) \tag{22}$$

Thus, the derivative is positive, as long as $\overset{0}{e}_x > \bar{x}$, vanishes at $x = \bar{x}$ and becomes negative thereafter. The maximum value of the area is therefore,

$$A(\bar{x}) = 2\bar{x}T_{\bar{x}} < \bar{x}T_0 \tag{23}$$

because of (17). Combining (19) and (23), the inequality

$$T_0^2 < 4\bar{x}T_{\bar{x}} = 4\bar{x}^2\ell_{\bar{x}} \tag{24}$$

is obtained, which can be rewritten as

$$\frac{T_0}{\bar{x}} < 2\sqrt{\ell_{\bar{x}}} \tag{25}$$

Noting from (17) that

$$2\ell_{\bar{x}}\bar{x} < T_0 \tag{26}$$

(25) can be combined with (26), to produce

$$2\ell_{\bar{x}} < \frac{T_0}{\bar{x}} < 2\sqrt{\ell_{\bar{x}}} \tag{27}$$

5. Average Age and Expectation of Life at Birth

Since $T_0 = \overset{0}{e}_0$ is the expectation of life at birth, (27) establishes an interesting inequality relationship between $\overset{0}{e}_0$ and the average age of the stationary population $\bar{x}$.

It may be mentioned in this context that usually, $\overset{0}{e}_0 > \bar{x}$ which, because of (27), requires that $\ell_{\bar{x}} > \frac{1}{2}$ (Table 1, cols 2, 4 and 5). However, it is also seen that the difference between $\overset{0}{e}_0$ and $\bar{x}$ decreases with $\overset{0}{e}_0$, and as could also be expected from the inequality relationship, $\ell_{\bar{x}}$ and $\overset{0}{e}_0$ are positively associated as well. It is therefore quite possible for expectations of life, lower than that shown in Table 1, to be less than $\bar{x}$. In fact, for the UN model life table (1956), level 0, the average age was found to be 23.5 years corresponding to the life expectancy of only 19.8 years and an

$\ell_{\bar{x}}$, as low as .37. Although, there are very few countries in the world today with a life expectancy that low, $\overset{0}{e}_0 > \bar{x}$ is at most, an empirical reality whereas, the actual mathematical relationship between the two can, at present, be expressed in the form of a boundary condition shown in (27). Further investigation is needed to find out the possibility or otherwise of improving this inequality relationship.

References

Keyfitz, N. & W. Flieger, 1971, Population: Facts and Methods of Demography, W.H. Freeman & Co.

Mitra, S., 1965, "Total Length of Life and Its Relation with the Parameters of Stable Population," Sankhyā, Vol. 27, Series B, pp. 189-192.

Mitra, S., 1973, "Optimum Values of Life Table Functions," Proceedings of the American Statistical Association, Social Statistics Section, pp. 356-358.

United Nations, 1956, Methods for Population Projections by Sex and Age, ST/SOA/ Series A, Population Studies, No. 25.

TABLE 1. Average Ages of Stationary Populations (m), Life Expectancies $\overset{0}{e}_0$, $\bar{x} = \overset{0}{e}_{\bar{x}}$, $\ell_{\bar{x}}$ and $T_{\bar{x}}$ for a Few Selected Countries (males only).

Country	$\overset{0}{e}_0 = T_0$	m	$\bar{x} = \overset{0}{e}_{\bar{x}}$	$\ell_{\bar{x}}$	$T_{\bar{x}}$
(1)	(2)	(3)	(4)	(5)	(6)
Cameron, 1964	34.2	28.0	27.6	.57	15.7
Indonesia, 1961 (incl. West Irian)	44.1	32.1	31.7	.66	20.9
S. Africa, 1961 (colored)	49.8	33.1	32.6	.74	24.1
Phillipines, 1960	55.4	35.7	35.2	.76	26.8
Chile, 1967	59.2	35.0	34.5	.84	29.0
Japan, 1966	68.5	36.6	36.0	.94	33.8
Denmark, 1967	70.7	37.4	36.8	.95	35.0